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Self-organized criticality in a sandpile model with threshold dissipation

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We study a nonconservative sandpile model in one dimension, in which, if the height at any site exceeds a threshold value, the site topples by transferring one particle along each bond connecting it to its neighbors. Its height is then set to 1, irrespective of the initial value. The model shows nontrivial critical behavior. We solve this model analytically in one dimension for all driving rates. We calculate all the two point correlation functions in this model, and find that the average local height decreases as the inverse of the distance from the nearest boundary and the power spectrum of fluctuations of the total mass varies as $1/f$.

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Self-organized criticality (SOC) was proposed by Bak, Tang, and Wiesenfeld [1] to explain the widespread occurrence of fractal structures and $1/f$ noise in nature. Since then several sandpilelike stochastic cellular automata models with threshold dynamics have been studied to understand the origin of criticality. In models with bulk mass conservation [2–5], the balance of rate of driving and the rate of loss of particles through the boundary leads to the divergence of the average avalanche size in the steady state. However, the origin of criticality in *nonconservative* sandpile models is still not well understood [6,7].

In this paper we study a discrete nonconservative sandpile model [8] in which mass dissipation (loss of a particle) occurs if the height at a site exceeds a threshold value. This threshold (for dissipation) is taken to be more than the threshold for toppling. As a result, dissipation does not occur in all topplings. The system self-organizes into a state in which most of the particles dissipate in the bulk and the loss through the boundary goes to zero in the thermodynamic limit. However, the model is still critical (see [9] for simulation results in two dimensions). We study this model analytically in one dimension. We find that in the steady state an interesting spatial structure emerges, which is described by a power law. We show that the power spectrum of the fluctuations of the total mass of the sandpile shows $1/f$ behavior, even for finite driving (unlike the forest fire model [10], which is critical only for infinitesimal driving). This is a nonconservative model of SOC, which we show to be critical for all driving rates.

The model is defined as follows: There is an integer height variable h_i (number of particles), at each site i. The particles are added to the system at randomly selected sites. If h_i exceeds the threshold height h_i^c, then h_i is set to 1 and one particle is transferred to each neighbor of i. This process occurs at all sites in parallel. We choose h_i^c to be equal to the coordination number for i inside the bulk and to be greater than the coordination number for i on the boundary. Note that this model is not Abelian [2]. In the Abelian sandpile model (ASM), h_i decreases by h_i^c after the toppling, whereas in this model h_i is set to 1 irrespective of its initial value. In other words, mass dissipation occurs if the height exceeds $h_i^d = h_i^c + 1$ (thresholds for dissipation).

To illustrate the difference between this model and the ASM, consider the model on a square lattice and choose an initial configuration in which the four corners of a given plaquette have all heights equal to 4. Chose $h_i^c = 4$ and $h_i^d = 5$. Now add a particle at the lower left corner of the plaquette, say at (i, j). First site (i, j) topples then sites $(i+1, j)$ and $(i, j+1)$ topple. After this the height at site $(i+1, j+1)$ becomes 6. So far there is no dissipation (just

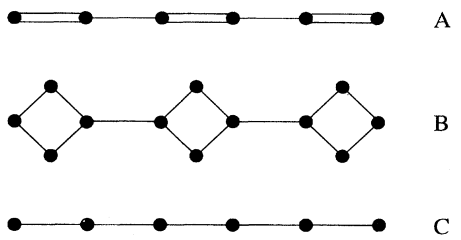

FIG. 1. One dimensional chains formed by joining (A) doublets, (B) diamonds, and (C) single sites.

as in the ASM). When site $(i+1, j+1)$ topples, unlike in the ASM, one particle is lost. Note that this model is the same as the ASM if the underlying graph does not have a loop.

The dynamics of this model resembles the dynamics of the continuous stick-slip model discussed by Feder and Feder (FF) [6]. The FF model has a real variable u_i at each site i, which grows slowly with time. If u_i exceeds U_c, then u_i is set to zero and u_j is increased by 1 for all neighbors j of site i.

In spite of the non-Abelian character, the set of recurrent configurations of this model S_1 is the same as that of the Abelian sandpile model S_2. To prove this, we first show $S_2 \subset S_1$. Let us add a particle in $C_0 \in S_2$ and let it relax by the rules of the ASM. Let the final configuration be C_1. Now we add a particle at the same site in C_0 and let it relax by the rules of this model. If there is no dissipation, then the final configuration is the same as C_1. If there is dissipation, then we compensate for the dissipation by adding particles one by one at sites where dissipation has occurred and letting the system relax. This process is repeated until no site is left where dissipation has occurred. Since in the ASM the final configuration does not depend on the order of the topplings, this sequence of additions and relaxations in this model must lead to C_1. Thus, in this model, there is a finite probability of transitions from one recurrent configuration of the ASM to the another. This implies that $S_2 \subset S_1$. One can easily check using the argument of Ref. [2] that the forbidden configurations of the ASM are also forbidden in this model. Therefore, $S_1 = S_2$. However, unlike the Abelian case, the probabilities of occurrence of different recurrent configurations in the steady state of this model need not be equal.

Let us analyze this model in one dimension. The model on a simple linear chain is not different from the ASM, because this lattice has no loops. Hence, we consider this model on decorated one dimensional chains, formed by joining unit cells which have loops (see Fig. 1). We have earlier studied ASM's on this type of chain and found interesting finite-size scaling behavior [11], which differs from that seen in simple linear chains [12].

Consider first the chain of doublets (case A, Fig. 1). We label the doublets by integers $i = -l$ to l and the sites inside the doublet by $j = 1$ to 2. The size of the chain $L = 2l+1$. We generalize the rule such that after the toppling one particle is transferred along each bond connecting the site to its neighbors, and take the threshold height equal to 3 for all sites. The recurrent configurations of this model are characterized using the burning algorithm as in the ASM. This algorithm is

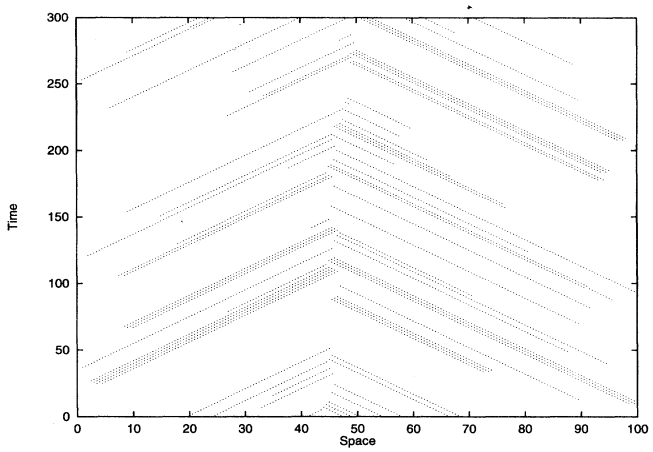

FIG. 2. Evolution of the sandpile. The dot denotes a toppling event.

specified by the following rule: A site is burnt if its height is greater than the number of bonds joining it to its unburnt neighbors (see [2], for details). A stable configuration is recurrent if and only if all the sites are eventually burnt.

In this algorithm the sites can be burnt in any order. The first site that can be burnt in a stable configuration is obviously a boundary site, because h_i^d is more than the number of bonds joining site i only if i is a boundary site. We let the burning start from the left boundary and hold the right boundary unburnt. The point at which burning from this boundary stops is called the break point (BP). Afterwards, the right boundary is burnt and subsequently the remaining sites. Thus the allowed values of (h_{i1}, h_{i2}) for i to the left of the BP are (3, 3), and (3, 2), and those for i to the right of the BP are (3, 3) and (2, 3). The allowed values at the BP are (1, 3), (3, 1), and (2, 3).

In any side the avalanche spreads to the first doublet that cannot be burnt from that side. For example, if the particle is added to the left of the BP the avalanche spreads to the BP on the right and to the first doublet of type (3, 2) on the left. For a typical avalanche the distance between the site at which the particle is added and the BP is $O(L)$. Therefore, the spread of the avalanche is $O(L)$.

When the avalanche crosses a doublet of type (3, 3), one particle is dissipated. Thus, each avalanche wipes out the (3, 3) type of doublets from a region of order L, leaving a small density of the (3, 3) type doublets. Since the steady state configurations are dominated by (3, 2) type doublets [(2, 3) type doublets] on the left [right] of the BP, the leftward [rightward] spread of the avalanche starting from the left [right] of the BP is small (see Fig. 2).

The BP shows an interesting stochastic dynamics. After the avalanche, it moves towards the starting point of the avalanche, by a distance of order 1. If i is the position of the BP, then the probability that an avalanche starts from the left of the BP is $(l+i)/(2L)$ to leading order [ignoring the density of the (3, 3) type of doublets], and that which starts from the right of the BP is $(l-i)/(2L)$. Hence, the mean displacement of the BP after the avalanche scales as $i/(2L)$. The fluctuation about this mean value is of order 1. The dynamics of the BP can thus be described by an equation of the following form:

$$\frac{dx}{dt} = -\frac{x}{2L} + \eta(t), \tag{1}$$

where x is the scaled variable i/L, and dt is of the order of the time interval at which avalanches hit the break point. The noise $\eta(t) \sim 1/L$, and is δ correlated. From the above equation it follows that the number of avalanches required to reach the steady state is of order L, and the asymptotic distribution of the BP goes as $\exp(-cx^2L)$, where c is a constant. The width of this distribution can be ignored in the large L limit. A typical avalanche then starts from the point at which the particle is added (source point) and terminates at the center. If the source point is i then the linear extension s and the duration t of an avalanche are equal to $|i-L/2|$. Averaging over i we get

$$\text{Prob}\ (X) \sim 2/L \quad \text{for} \quad X < L/2, \tag{2}$$

and 0 otherwise, where $X = t,s$.

Now we calculate the correlation functions in this model. The model has two intrinsic time scales, the time taken for one toppling (defined as one time step), and the time interval T between two consecutive additions of particles. Let us first consider the case $T > L$ such that there is no overlap between two consecutive avalanches [see Eq. (2)].

It is convenient to use a two state variable s_i, because there are only two allowed configurations of each doublet (except the break point). For the (3, 3) type doublet, $s_i = 1$, otherwise $s_i = 0$. Consider the evolution s_i for $\sqrt{L} \ll i \ll l$. We can assume that the BP is on the left of i. If the avalanche starts from the right of i, then it crosses i to reach the BP, setting $s_i = 0$. The probability of this transition goes as $(l-i)/(2L)$. The transition of s_i from 0 to 1 occurs if the particle is added at the left site of the ith doublet, or if it is added to the left of the ith doublet, but the avalanche reaches the $(i-1)$th doublet so that the left site of the ith doublet receives a particle. The probability of this transition goes as $1/L$. The corresponding probabilities for $-l \ll i \ll -\sqrt{L}$ can be obtained by replacing i by $(-i)$ and l by $(-l)$. Thus the probability $P_n(i)$, that $s_i = 1$ after additions of n particles, satisfies the following equation (to leading order):

$$P_{n+1}(i) = \frac{1}{L}[1 - P_n(i)] - \left(1 - \frac{r(i)}{2L}\right) P_n(i), \tag{3}$$

where $r(i)$ is the distance of i from the nearest boundary. A straightforward calculation using the above equation gives

$$\langle s_i \rangle \sim 2/r(i), \tag{4}$$

and the autocorrelation

$$\langle s_i(t_0)\ s_i(t_0+t) \rangle \sim \frac{1}{r(i)} (1 - \tau_i^{-1})^{t/T}, \tag{5}$$

for $t \gg T$, where $\tau_i = 2L/r(i)$ and $\langle \cdots \rangle$ denotes the average over t_0. The value of s_i at time t is denoted by $s_i(t)$. Note that the correction to Eq. (3) coming from the nonzero density of s_i[Eq. (4)] vanishes in the large L limit. The steady state of this model is not translationally invariant, but is self-similar in space, i.e., the density and the relaxation time τ_i depend on its distance from the nearest boundary as a power law. In Fig. 3, we plot the density of s_i against the distance from the left boundary. Note the excellent agreement of our numerical data with Eq. (4).

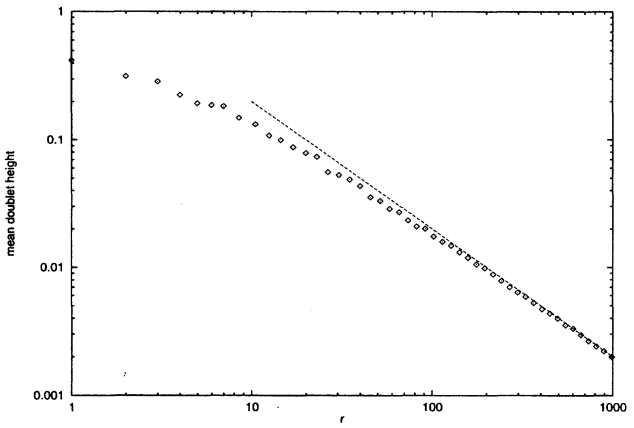

FIG. 3. Log-log plot of the mean height of a doublet (minimum height is subtracted) versus the distance from the nearest boundary r. The dashed line shows the analytic expression.

To calculate the correlation between $s_i(t)$ and $s_j(t')$, consider the evolution of the joint probability of s_i and s_j. There are four possible values of (s_i, s_j). Let the four-vector, $P_n(i,j)$, denote the joint probability of (s_i, s_j) after the addition of n particles. Its evolution can be described by the following linear equation (to leading order):

$$P_{n+1}(i,j) = G(i,j) P_n(i,j), \tag{6}$$

where $G(i,j)$ is the (4×4) matrix giving the transition probabilities of s_i and s_j. This can be calculated in exactly the same way as the transition probability of s_i calculated above. For example, (1,1) goes to (0, 0) if the avalanche passes over both the doublets i and j. If $\sqrt{L} \ll i \ll j \ll l$, then the probability of this transition goes as $r(j)/(2L)$. Other transition probabilities can also be calculated in the same way. From Eq. (6) one can find the steady state correlation functions. For i and j on the same side of the center of the chain

$$\langle s_i s_j \rangle_c \sim \frac{1}{r(i)^2}, \tag{7}$$

$$\langle s_i(t_0)\ s_j(t_0+t) \rangle_c \sim \frac{1}{r(i)^2} (1 - \tau_j^{-1})^{t/T} \quad \text{for} \quad r(i) > r(j), \tag{8}$$

$$\langle s_i(t_0)\ s_j(t_0+t) \rangle_c \sim -\frac{1}{r(j)[r(j)-r(i)]^2} \times (1 - \tau_i^{-1})^{t/T} \quad \text{for} \quad r(i) < r(j). \tag{9}$$

The subscript c refers to the connected part of the correlation function. If the i and j are on different sides of the center then the correlation can be ignored. Note that, in Eq. (8), the first factor gives the equal time correlation, and the second factor gives the relaxation of s_j. In Eq. (9) the $s_i(t_0)$ and $s_j(t_0+t)$ are anticorrelated because an avalanche which makes $s_i = 1$ sweeps over site j, setting $s_j = 0$.

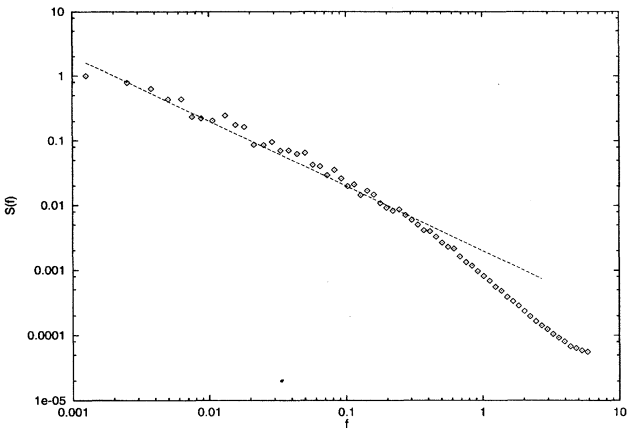

FIG. 4. Log-log plot of the power spectrum of total mass fluctuation $S(f)$ vs frequency f. For comparison the theoretical curve is shown as the dashed line.

The autocorrelation of mass (sum of heights) can be obtained using Eqs. (5), (8), and (9). The power spectrum of mass fluctuation $S(f)$ is obtained by taking the real part of the Fourier transform of the autocorrelation

$$S(f) \sim \frac{1}{f} \quad \text{for} \quad \frac{2\pi}{TL} \ll f \ll 2\pi. \tag{10}$$

In Fig. 4, we show the log-log plot of the numerical value of $S(f)$ versus f. This is obtained by taking the square of the Fourier transform of the time sequence of length 5000, for a chain of size 5000. We find that at low frequencies the data agree with the theoretically predicted $1/f$ behavior. The scaling region increases with the increase in the lattice size. Since we have taken only the slowest relaxation mode in Eqs. (8), (9), and (5), the agreement is not good in the high frequency regime.

Consider now the case in which particles are added at a faster rate, i.e., $1 \ll T \ll L$. We note that the interaction between the avalanches can be ignored because the avalanches propagate with the same velocity towards the BP (see Fig. 2). Thus the steady state properties of the model as described by Eqs. (7) and (4) and the autocorrelation given in Eq. (5) remain unchanged in this case. Since the avalanche takes $2|j-i|$ time steps to travel from i to j, there is no anticorrelation between $s_i(t_0)$ and $s_j(t_0+t)$ for $t<2|j-i|$[see Eq. (9)]. However, the contribution to the mass autocorrelation coming from this term can be ignored for $|i-j|>\sqrt{L}$, or for $t>\sqrt{L}$. Thus for $f<2\pi T/\sqrt{L}$, the power spectrum $S(f)\sim 1/f$.

To check the robustness of our results we have studied the model on the diamond chain (case B in Fig. 1). An avalanche in this case has the same structure as in case A, i.e., it spreads to the BP on one side, to a distance of $O(1)$ on the other side, and the BP is confined in a region of $O(\sqrt{L})$. As a result, the power laws of the average height and the power spectrum of mass fluctuation are also the same.

The two special doublet configurations in this model, which stop the avalanches, are reminiscent of the "trough" and "trap" of the "singular diffusion" type model studied by Carlson *et al.* [5]. In that case too there is a power law decay of the height profile that follows from the singularity of the effective diffusion coefficient of the particles as a function of the coarse grained density [4,13]. However, in our model the mechanism of self-organization is different as the diffusion coefficient does not diverge.

To summarize, we have determined exactly the critical behavior of a non-Abelian nonconservative sandpile model in one dimension. The critical steady state shows spatial structures. We determine time dependent correlation functions for finite driving rate, and show that fluctuations of the mass of the sandpile have a $1/f$ spectrum.

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[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).

[2] D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990).

[3] S. N. Majumdar and D. Dhar, Physica A **185**, 129 (1992).

[4] A. B. Chhabra, M. J. Feigenbaum, L. P. Kadanoff, A. J. Kolan, and I. Procaccia, Phys. Rev. E **47**, 3099 (1993).

[5] J. Carlson, J. Chayes, E. R. Grannan, and G. H. Swindle, Phys. Rev. Lett. **65**, 2547 (1990).

[6] H. J. S. Feder and J. Feder, Phys. Rev. Lett. **66**, 2669 (1991).

[7] Z. Olami, H. J. S. Feder, and K. Christensen, Phys. Rev. Lett. **64** 1927 (1990); K. Christensen and Z. Olami, Phys. Rev. A **46**, 1829 (1992); Peter Grassberger, Phys. Rev. E **49**, 2436 (1994); A. A. Middleton and C. Tang, Phys. Rev. Lett. **74**, 742 (1995), and references therein.

[8] We learned of this model from J. Krug (private communication), who also informed us that this model had been studied earlier by M. S. Bourzutschky and C. H. Bennett (unpublished).

[9] M. S. Bourzutschky, Ph.D. thesis, California Institute of Technology, 1993 (unpublished); D. Dhar, J. Krug, S. S. Manna, and E. R. Speer (unpublished).

[10] B. Drossel and F. Schwabl, Phys. Rev. Lett. **69**, 1629 (1992); B. Drossel, S. Clar, and F. Schwabl, *ibid.* **71**, 3739 (1993); P. Grassberger, J. Phys. A **26**, 2081 (1993).

[11] A. A. Ali and D. Dhar, Phys. Rev. E **51**, R2705 (1995).

[12] P. Ruelle and S. Sen, J. Phys. A **25**, L1257 (1992).

[13] J. Krug, Phys. Rev. E **47**, 730 (1993).